What Are Numbers?

A philosophical search for their true nature

Based mainly on an article from Paul Benacerraf and supported by various articles by Michael Resnik, Mark Balaguer and Hartry Field.

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A very old question

Renewed attention of philosophers in modern times due to ever-increasing understanding of mathematics.

What is a number?

What objects do number-words ('one', 'two', etc.) **refer** to?

Gottlob Frege

$3 \stackrel{?}{=} \text{Gaius Iulius Caesar}$

What does this refer to?

This name refers to the emperor of the Roman Empire

A Long Time Ago in Ancient Greece…

~~0~~, 1, 2, 3, 4, 5, …

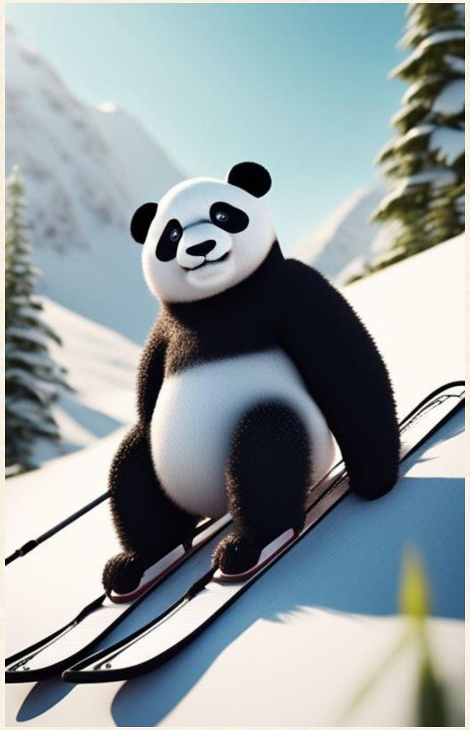
- 0 → Does not exist!
- 1 → Ist not a number, but a unit length!
- 2, 3, 4, 5, … → Multitudes are truly numbers!

❖ Definition of numbers conditional to unit size (length, area, …)
❖ Exercises had no variables but number-valued solutions

What might a mathematician answer?

The average mathematician

You



What is 3?

The number after 2!

And what is 2?

The number after 0 and 1!

And what is 0?

The smallest number!

Definition of numbers by axiomatic, recursive construction!
Numbers are sets!

A story of two children…

Ernie

Bert

Ernie and Bert receive a very formal education in Logic

- In particular: Set Theory
- Their parents then need only to point out what part of what they already know is what ordinary people call “numbers”

There is a set whose members ordinary people refer to as the (natural) numbers:

- The „numbers“ are the infinite set $\mathbb{N}$: 1 and successors (by recursion)
- The less-than relation „<“, they knew as „R“
- The operators „+“, etc. as set operations

The extra-mathematical use of numbers in the real world: Counting

- Intransitive → Ordinal Numbers
- Transitive → Cardinal Numbers

Delighted with what they have learned, they start proving theorems about numbers!

What is happening?
Ernie and Bert are fighting!

$3 \in 17!$

$3 \notin 17!$

Ernie: $\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$

Bert: $\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$

Who is right?

The accounts differ at places where there is no connection whatsoever between features of the accounts and our uses of the numbers and number words.

⇒ If it is possible at all to identify numbers as some universal entity, then it is not sets!

Does the question even make sense?

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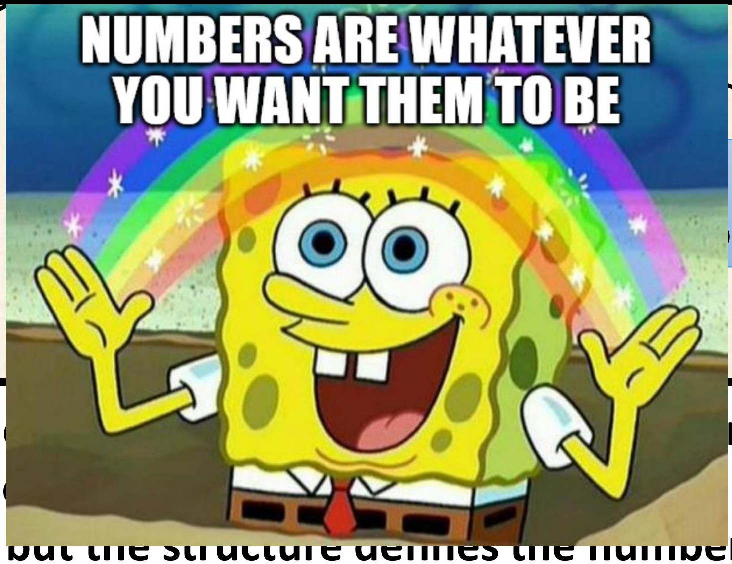

If any recursive progressi rs, the condition lies not on the *objects* (sets) but
⇒ **Not any individuality, but the structure defines the numbers!**

„“Objects” do not do the job of numbers singly; the whole system performs the job or nothing does. […] The pointlessness of trying to determine which objects the numbers are thus derives from the pointlessness of asking the question of any individual number. – *Paul Benacerraf*

BACKUP: Why does R need to be recursive?

- Given two numbers, we need to be able to find out in a finite amount of steps, which number is greater
- Now, imagine the progression $\mathrm{C} = a_1, b_1, a_2, b_2, \ldots$ with a_n being the sequence of Gödel numbers of valid formulas of quantification theory, under some suitable numbering and b_n being the sequence of positive numbers that are not Gödel numbers of valid formulas.
- This expression is completely well-defined but unusable since we cannot in a finite number of steps calculate its elements. There are Gödel numbers that correspond to formulas that are correct but can never be proven to be.